Reports

How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension

Abstract. *Geographical curves are so involved in their detail that their lengths are often infinite or, rather, undefinable. However, many are statistically "self-similar," meaning that each portion can be considered a reduced-scale image of the whole. In that case, the degree of complication can be described by a quantity* D *that has many properties of a "dimension," though it is fractional; that is, it exceeds the value unity associated with the ordinary, rectifiable, curves.*

Seacoast shapes are examples of highly involved curves such that each of their portion can—in a statistical sense —be considered a reduced-scale image of the whole. This property will be referred to as "statistical self-similarity." To speak of a length for such figures is usually meaningless. Similarly (*1*), "the left bank of the Vistula, when measured with increased precision, would furnish lengths ten, hundred or even thousand times as great as the length read off the school map." More generally, geographical curves can be considered as superpositions of features of widely scattered characteristic size; as ever finer features are taken account of, the measured total length increases, and there is usually no clearcut gap between the realm of geography and details with which geography need not be concerned.

Quantities other than length are thus needed to discriminate between various degrees of complication for a geographical curve. When a curve is self-similar, it is characterized by an exponent of similarity, *D*, which possesses many properties of a dimension, though it is usually a fraction greater than the dimension 1 commonly attributed to curves. We shall reexamine in this light some empirical observations by Richardson (*2*). I propose to interpret them as implying, for example, that the dimension of the west coast of Great Britain is $D = 1.25$. Thus, the so far esoteric concept of "random figure of fractional dimension" is shown to have simple and concrete applications and great usefulness.

Self-similarity methods are a potent tool in the study of chance phenomena, including geostatistics, as well as economics (*3*) and physics (*4*). In fact, many noises have dimensions *D* contained between 0 and 1, so that the scientist ought to consider dimension as a continuous quantity ranging from 0 to infinity.

Returning to the claim made in the first paragraph, let us review the methods used when attempting to measure the length of a seacoast. Since a geographer is unconcerned with minute details, he may choose a positive scale *G* as a lower limit to the length of geographically meaningful features. Then, to evaluate the length of a coast between two of its points *A* and *B*, he may draw the shortest inland curve joining *A* and *B* while staying within a distance *G* of the sea. Alternatively, he may draw the shortest line made of straight segments of length at most *G*, whose vertices are points of the coast which include *A* and *B*. There are many other possible definitions. In practice, of course, one must be content with approximations to shortest paths. We shall suppose that measurements are made by walking a pair of dividers along a map so as to count the number of equal sides of length *G* of an open polygon whose corners lie on the curve. If *G* is small enough, it does not matter whether one starts from *A* or *B*. Thus, one obtains an estimate of the length to be called $L(G)$.

Unfortunately, geographers will disagree about the value of *G*, while $L(G)$ depends greatly upon *G*. Consequently, it is necessary to know $L(G)$ for several values of *G*. Better still, it would be nice to have an analytic formula linking $L(G)$ with *G*. Such a formula, of an entirely empirical character, was proposed by Lewis F. Richardson (*2*) but unfortunately it attracted no attention. The formula is $L(G) = M\,G^{1-D}$, where *M* is a positive constant and *D* is a constant at least equal to unity. This

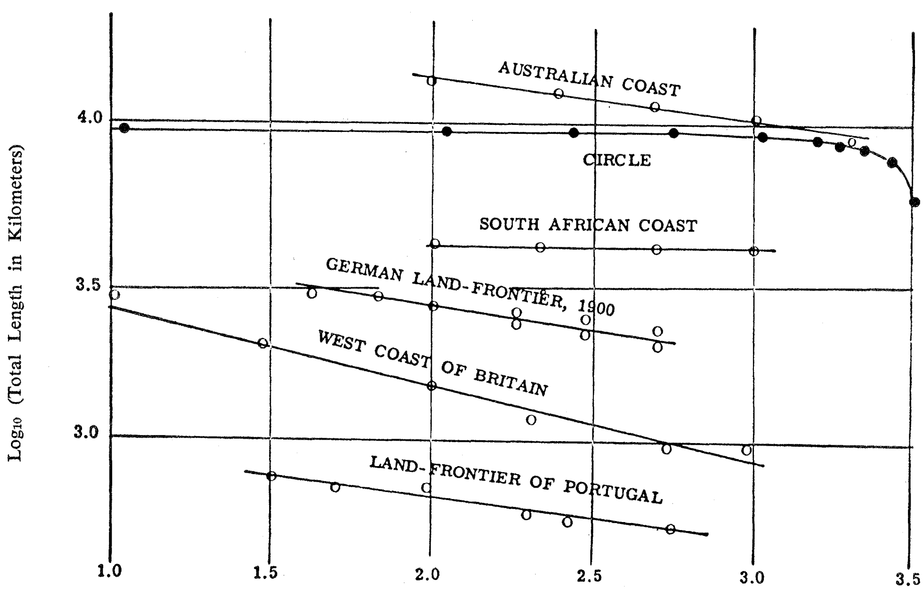

Fig. 1. Richardson's data concerning measurements of geographical curves by way of polygons which have equal sides and have their corners on the curve. For the circle, the total length tends to a limit as the side goes to zero. In all other cases, it increases as the side becomes shorter, the slope of the doubly logarithmic graph being in absolute value equal to D–1. (Reproduced from *2*, Fig. 17, by permission.)

D, a "characteristic of a frontier, may be expected to have some positive correlation with one's immediate visual perception of the irregularity of the frontier. At one extreme, $D = 1.00$ for a frontier that looks straight on the map. For the other extreme, the west coast of Britain was selected because it looks like one of the most irregular in the world; it was found to give $D = 1.25$. Three other frontiers which, judging by their appearance on the map were more like the average of the world in irregularity, gave $D = 1.15$ for the land frontier of Germany in about A.D. 1899; $D = 1.14$ for the land frontier between Spain and Portugal and $D = 1.13$ for the Australian coast. A coast selected as looking one of the smoothest in the atlas, was that of South Africa and for it, $D = 1.02$."

Richardson's empirical finding is in marked contrast with the ordinary behavior of smooth curves, which are endowed with a well-defined length and are said to be "rectifiable." Thus, to quote Steinhaus (*1*) again, "a statement nearly adequate to reality would be to call most arcs encountered in nature not rectifiable. This statement is contrary to the belief that not rectifiable arcs are an invention of mathematicians and that natural arcs are rectifiable: it is the opposite that is true."

I interpret Richardson's relation as contrary to the belief that curves of dimension greater than one are an invention of mathematicians. For that, it is necessary to review an elementary feature of the concept of dimension and to show how it naturally leads to the consideration of fractional dimensions.

To begin, a straight line has dimension one. Hence, for every positive integer N, the segment $(0 \leqslant x < X)$ can be exactly decomposed into N nonoverlapping segments of the form $[(n\text{-}1)X/N \leqslant x < nX/N]$, where n runs from 1 to N. Each of these parts is deducible from the whole by a similarity of ratio $r(N) = 1/N$. Similarly, a plane has dimension two. Hence, for every perfect square N, the rectangle $(0 \leqslant x < X; 0 \leqslant y < Y)$ can be decomposed exactly into N nonoverlapping rectangles of the form $[(k\text{-}1)X/\sqrt{N} \leqslant x < kX/\sqrt{N}; (h\text{-}1)Y/\sqrt{N} \leqslant y < hY/\sqrt{N}]$, where k and h run from 1 to $\sqrt{N}$. Each of these parts is deducible from the whole by a similarity of ratio $r(N) = 1/\sqrt{N}$. More generally, whenever $N^{1/D}$ is a positive integer, a D-dimensional rectangular parallelepiped can be decomposed into N parallelepipeds deducible from the whole by a similarity of ratio $r(N) = 1/N^{1/D}$. Thus, the dimension D is characterized by the relation $D = -\log N/\log r(N)$.

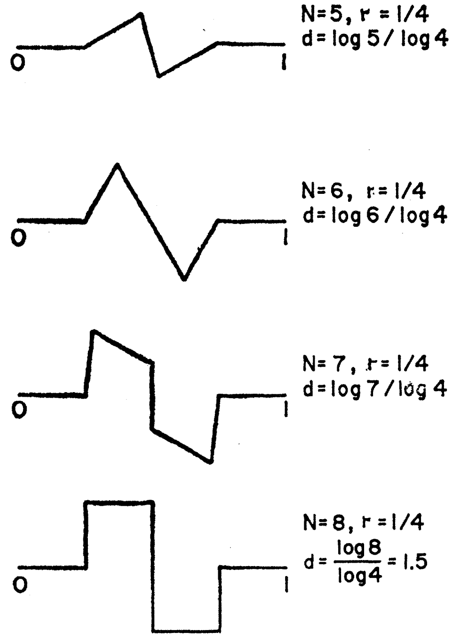

Fig. 2. Nonrectifiable self-similar curves can be obtained as follows. Step 1: Choose any of the above drawings. Step 2: Replace each of its N legs by a curve deduced from the whole drawing through similarity of ratio 1/4. One is left with a curve made of N^2 legs of length $(1/4)^2$. Step 3: Replace each leg by a curve obtained from the whole drawing through similarity of ratio $(1/4)^2$. The desired self-similar curve is approached by an infinite sequence of these steps.

This last property of the quantity D means that it can also be evaluated for more general figures that can be exactly decomposed into N parts such that each of the parts is deducible from the whole by a similarity of ratio $r(N)$, or perhaps by a similarity followed by rotation and even symmetry. If such figures exist, they may be said to have $D = -\log N/\log r(N)$ for dimension (*5*). To show that such figures exist, it suffices to exhibit a few obvious variants of von Koch's continuous nondifferentiable curve. Each of these curves is constructed as a limit. Step 0 is to draw the segment (0, 1). Step 1 is to draw either of the kinked curves of Fig. 2, each made up of N intervals superposable upon the segment (0, ¼). Step 2 is to replace each of the N segments used in step 1 by a kinked curve obtained by reducing the curve of step 1 in the ratio $r(N) = ¼$. One obtains altogether N^2 segments of length 1/16. Each repetition of the same process adds further detail; as the number of steps grows to infinity, our kinky curves tend toward continuous limits and it is obvious by inspection that these limits are self-similar, since they are exactly decomposable into N parts deducible from the whole by a similarity of ratio $r(N) = ¼$ followed by translation. Thus, given N, the limit curve can be said to have dimension $D = -\log N/\log r(N) = \log N/\log 4$. Since N is greater than 4 in our examples, the corresponding dimensions all exceed unity. Let us now consider length: at step number s, our approximation is made of N^s segments of length $G = (¼)^s$, so that $L = (N/4)^s = G^{1-D}$. Thus, the length of the limit curve is infinite, even though it is a "line." (Note that it is not excluded for a plane curve to have a dimension equal to 2. An example is Peano's curve, which fills up a square.)

Practical application of this notion of dimension requires further consideration, because self-similar figures are seldom encountered in nature (crystals are one exception). However, a statistical form of self-similarity is often encountered, and the concept of dimension may be further generalized. To say that a (closed) plane figure is chosen at random implies several definitions. First, one must select a family of possible figures, usually designated by Ω. When this family contains a finite number of members, the rule of random choice is specified by attributing to each possible figure a well-defined probability of being chosen. However, Ω is in general infinite and each figure has a zero probability of being chosen. But positive probabilities can be attached to appropriately defined "events" (such as the event that the chosen figure differs little—in some specified sense—from some specified figure).

For the family Ω, together with the definition of events and their probabilities, to be self-similar, two conditions are needed. First, each of the possible figures must be constructible by somehow stringing together N figures, each of which is deduced from a possible figure by a similarity of ratio r; second, the probabilities must be so specified that the same value is obtained whether one selects the overall figure at one swoop or as a string. (The value of N may either be arbitrary, or chosen from some specific sequence, such as the perfect squares relative to nonrandom rectangles, or the integral powers of 4, 5, 6, or 7 encountered in the curves built as in Fig. 2.) In case that the value of r is specified by choosing N, one can consider $-\log N/\log r$ a similarity dimension. More usually, however, given r, N will take different values for different figures of Ω. As one considers points "sufficiently far" from each other, the details on a "sufficiently fine" scale may become asymptotically independent, in such a way that $-\log N/\log r$ almost surely tends to some limit as r tends to zero. In that case, this limit may be considered a similarity dimension. Under wide conditions, the length of approximating polygons will asymptotically behave like $L(G) \sim G^{1-D}$.

To specify the mathematical conditions for the existence of a similarity dimension is not a fully solved problem. In fact, even the idea that a geographical curve is random raises a number of conceptual problems familiar in other applications of randomness. Therefore, to return to Richardson's empirical law, the most that can be said with perfect safety is that it is compatible with the idea that geographical curves are random self-similar figures of fractional dimension D. Empirical scientists having to be content with less than perfect inductions, I favor the more positive interpretation stated at the beginning of this report.

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References and Notes

1. H. Steinhaus, *Colloquium Math.* **3**, 1 (1954), where earlier references are listed.
2. L. F. Richardson, in *General Systems Yearbook* **6**, 139 (1961).
3. B. Mandelbrot, *J. Business* **36**, 394 (1963), or in *The Random Character of Stock Market Prices*, P. H. Cootner, Ed. (M.I.T. Press, Cambridge, Mass, 1964), p. 297.
4. B. Mandelbrot, *IEEE Inst. Elect. Electron. Eng. Trans. Commun. Technol.* **13**, 71 (1965) and *IEEE Inst. Elect. Electron. Eng. Trans. Inform. Theory* **13** (1967). Very similar considerations apply in turbulence, where the characteristic sizes of the "features" (that is, the eddies) are also very widely scattered, as was first pointed out by Richardson himself in the 1920's.
5. The concept of "dimension" is elusive and very complex, and is far from exhausted by the simple considerations of the kind used in this paper. Different definitions frequently yield different results, and the field abounds in paradoxes. However, the Hausdorff-Besicovitch dimension and the capacitary dimension, when computed for random self-similar figures, have so far yielded the same value as the similarity dimension.

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Zonation of Uplifted Pleistocene Coral Reefs on Barbados, West Indies

Abstract. *The coral species composition of uplifted Pleistocene reefs on Barbados is very similar to Recent West Indian reefs.* Acropora palmata, Acropora cervicornis, *and* Montastrea annularis *are quantitatively the most important of the coral species.*

The island of Barbados in the West Indies emerged throughout the Pleistocene. During this emergence, fringing and barrier reefs formed from time to time around the island and resulted in the formation of a terraced "coral cap." Eighteen major reef tracts of different relative ages have been defined, the older reefs being located at the higher elevations (*1*).

Thirty road cuts through the reef terraces were examined to study the coral composition of these reefs. On traverses made perpendicular to the reef tract trends, from the base up to the crest of the terraces, a zonation of corals—largely in growth position—can be observed (Fig. 1). The zonation is composed of four major elements: (i) the coral-head zone, (ii) the *Acropora cervicornis* zone, (iii) the *Acropora palmata* zone, and (iv) the rear zone.

A distinct advantage in the study of uplifted reefs is that the deep-water communities can be examined in as great detail as those of the reef crest. Exposures at the base of the terraces are composed largely of coral species which tend to form massive coral heads. At several localities, *Montastrea annularis* makes up as much as 50 percent of the total exposure. However, at other localities, *M. annularis* composes only 10 to 15 percent of the total exposure and *Siderastrea siderea, Siderastrea radians, Diploria strigosa*, and *Diploria labyrinthiformis* are equally important. Numerous other coral species are also present in the coral-head zone, but they rarely exceed 5 percent of the total exposure. These include *Porites astreoides, Agaricia agaricites, Favia fragum, Meandrina meandrites, Meandrina brazilensis, Colpophyllia natans, Montastrea cavernosa,* and more delicately branched types such as *Porites porites, Eusmilia fastigiata,* and *Madracis* sp. The total coral content of some exposures in the coral-head zone exceeds 50 percent with the remainder being infilling matrix. The individual species occur in clumps with first one species being important and then another. Coralline algae are not abundant in this zone but at times form thick crusts on the upper surfaces of the coral heads.

Moving upward and back into the reef terrace, the zone of coral heads gives way gradually to a zone composed almost exclusively of *Acropora cervicornis* (Fig. 3). At times, *A. cervicornis* makes up as much as 75 to 80 percent of the total exposure. The average diameter of the individual branches is 2.0 to 2.5 cm although diameters in excess of 5 cm have been noted. Coralline algae often form thin coatings on the branches.

Near the upper portions of the *A. cervicornis* zone, *Montastrea annularis* along with *Siderastrea* sp. and *Diploria* sp. frequently are seen scattered in among the rich *A. cervicornis.* The *M. annularis* here has the growth form of heads and tall, thin columns or "pipes." At a few localities, thick developments (6 to 9 m) of *M. annularis* heads and pipes occur at the position normally occupied by the upper portions of the *A. cervicornis* zone. Such occurrences appear to be analogous to the "buttress zone" reported for Recent West Indian reefs along the north coast of Jamaica (*2*).

Near the crest of the reef terraces, the *A. cervicornis* zone grades rather rapidly into a zone which is composed almost exclusively of *Acropora palmata* (Fig. 4). Some of the reef terraces on Barbados, however, do not show the development of an *A. palmata* zone. The *A. palmata* increases in size and overall abundance towards the central portions of the zone where it reaches a maximum development, often making up to 70 percent of the exposure. For the reef terraces examined, the zone averages 75 m wide. Coralline algae are most abundant in this zone, and they often form heavy crusts on the massive branches of *A. palmata.* The algal crusts are particularly thick on the upper surface of the branches with crusts 5 to 8 cm thick not being unusual.

Moving back from the crest of the reef terrace, *A. palmata* gradually decreases in importance and is replaced by a wide variety of corals. This variety is comparable to that found in the